THE CHINESE UNIVERSITY OF HONG KONG
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MATH 2055 Tutorial 2 (Sep 23)

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1. Prove $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Solution:
$\forall$ integer $m > 2$, it follows that $\frac{2}{m} < 1$
(to avoid abusing index, we use another index)
$\forall \epsilon > 0$,
$\forall n > max\{\frac{4}{\epsilon}, 3\}$,

$$\begin{aligned}\left|\frac{2^n}{n!} - 0\right| &= \frac{2^n}{n!} \\ &= \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{n}\right) \\ &< 2\left(\frac{2}{n}\right) \\ &< \epsilon\end{aligned}$$

$\therefore \lim_{n\to\infty} \frac{2^n}{n!} = 0$. □

2. Prove that $\lim_{n\to\infty} \prod_{i=1}^{n} \left(\frac{i - \frac{1}{2}}{i}\right) = 0$

> Remark:
> Here the notation $\prod_{i=1}^{n}$ means 'the product of "something indexed by i" for i from 1 to n. E.g.
>
> $$\prod_{i=1}^{n} 2^i \quad \text{means} \quad 2^1 \times 2^2 \times 2^3 \times \cdots \times (2^n).$$

Solution:
$\forall$ integer $m > 1$,

$$\begin{aligned}\frac{\sqrt{(m-1)(m+1)}}{m} &= \frac{\sqrt{m^2-1}}{m} \\ &< \frac{\sqrt{m^2}}{m} \qquad (*) \\ &= 1\end{aligned}$$

$\forall\ \epsilon > 0$,
$\forall$ integer $n > max\{\frac{1}{2\epsilon^2}, 1\}$,

$$\begin{aligned}\left|\prod_{i=1}^{n}\frac{i-\frac{1}{2}}{i} - 0\right| &= \prod_{i=1}^{n}\frac{2i-1}{2i}\\ &= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2n-3}{2n-2}\right)\left(\frac{2n-1}{2n}\right)\\ &= \left(\frac{\sqrt{1\times 3}}{2}\right)\left(\frac{\sqrt{3\times 5}}{4}\right)\cdots\left(\frac{\sqrt{(2n-3)(2n-1)}}{2n-2}\right)\left(\frac{\sqrt{2n-1}}{2n}\right)\\ &< \frac{\sqrt{2n-1}}{2n} \qquad by(*)\\ &< \frac{1}{\sqrt{2n}}\\ &< \epsilon\end{aligned}$$

$\therefore \lim_{n\to\infty} \prod_{i=1}^{n} \frac{i-\frac{1}{2}}{i} = 0.$ □

3. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by
$w_n = mid\{x_n, y_n, z_n\}$ is also convergent.

(definition: $mid\{a, b, c\} = b$ if $a \leq b \leq c$) (i.e. picking the middle term).

<u>Solution</u>:
WLOG, assume $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$,
and $x \leq y \leq z$.

$\forall\ \epsilon > 0$,

(Case 1). $x < y < z$,

take $d = \dfrac{min\{z-y, y-x\}}{3}$, (to separate the 3 sequences)
$\because \lim_{i\to\infty} x_i = x$,

$\therefore \exists N_1$ such that $\forall i > N_1$, $|x_i - x| < d$

$\because \lim_{j\to\infty} y_j = y$,

$\therefore \exists N_2$ such that $\forall j > N_2$, $|y_j - y| < mid\{d, \epsilon\}$

$\because \lim_{k\to\infty} z_k = z$,

$\therefore \exists N_3$ such that $\forall k > N_3$, $|z_k - z| < d$

take $N = max\{N_1, N_2, N_3\}$, $\forall n > N$,

$z_n > z - d > y + d > y_n > y - d > x + d > x_n$,

$\therefore w_n = y_n$

$|w_n - mid\{x, y, z\}| = |y_n - y| < \epsilon$

$\therefore \lim_{n\to\infty} w_n = mid\{x, y, z\}$

(Case 2). if $x = y < z$ or $x < y = z$,

By replacing d with $\frac{z-y}{3}$ or $\frac{y-x}{3}$ in each case.

(need a complete argument in exam)

(Case 3). if $x = y = z$, no need to introduce d. □

4. (a) Let $\lim_{n\to\infty} x_n = x$ and x_n are positive. Let $g_n = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$, prove that $\lim_{n\to\infty} g_n = x$.

<u>Solution</u>:

(case 1). $x > 0$,

let ϵ which $\frac{3x}{2} > \epsilon > 0$, (WLOG, we can asuume ϵ small enough,which is used for finding lower bound).

$\because \lim_{i\to\infty} x_i = x$,

$\therefore \exists N_1$ such that $\forall i > N_1$, $|x_i - x| < \frac{\epsilon}{3}$.

As $\lim_{m\to\infty} [x + \frac{\epsilon}{3}]^{\frac{m-N_1}{m}} = x + \frac{\epsilon}{3}$,

$\therefore \exists N_2$, such that $\forall m > N_2$, $\left|[x + \frac{\epsilon}{3}]^{\frac{m-N_1}{m}} - [x + \frac{\epsilon}{3}]\right| < \frac{\epsilon}{3}$.

As $\lim_{p\to\infty} [x - \frac{\epsilon}{3}]^{\frac{p-N_1}{p}} = x - \frac{\epsilon}{3}$,

$\therefore \exists N_3$, such that $\forall p > N_3$, $\left|[x - \frac{\epsilon}{3}]^{\frac{p-N_1}{p}} - [x - \frac{\epsilon}{3}]\right| < \frac{\epsilon}{3}$.

Take $c = \prod_{j=1}^{N_1} x_j$,

as $\lim_{k\to\infty} c^{\frac{1}{k}} = 1$,

$\therefore \exists N_4$, such that $\forall k > N_3$, $\left|c^{\frac{1}{k}} - 1\right| < \dfrac{(\frac{\epsilon}{3})}{x + \frac{2\epsilon}{3}}$

$\Longrightarrow c^{\frac{1}{k}} < \dfrac{x+\epsilon}{x+\frac{2\epsilon}{3}}.$

As $\lim\limits_{p\to\infty} c^{\frac{1}{p}} = 1,$

$\therefore\ \exists N_5$, such that $\forall p > N_4$, $\left|c^{\frac{1}{p}} - 1\right| < \dfrac{\left(\frac{\epsilon}{3}\right)}{x-\frac{2\epsilon}{3}}$

$\Longrightarrow \dfrac{x-\epsilon}{x-\frac{2\epsilon}{3}} < c^{\frac{1}{p}}.$

$\forall\ n\ >\ max\{N_1, N_2, N_3, N_4, N_5\},$

$$\begin{aligned} g_n - x &= \left(c^{\frac{1}{n}}\right)\left(\prod_{j=N_1+1}^{n} x_j\right)^{\frac{n-N_1}{n}} - x \\ &< \left(c^{\frac{1}{n}}\right)\left[x+\frac{\epsilon}{3}\right]^{\frac{n-N_1}{n}} - x \\ &< \left(c^{\frac{1}{n}}\right)\left(x+\frac{2\epsilon}{3}\right) - x \\ &< \epsilon \end{aligned}$$

$$\begin{aligned} g_n - x &= \left(c^{\frac{1}{n}}\right)\left(\prod_{j=N_1+1}^{n} x_j\right)^{\frac{n-N_1}{n}} - x \\ &> \left(c^{\frac{1}{n}}\right)\left[x-\frac{\epsilon}{3}\right]^{\frac{n-N_1}{n}} - x \\ &> \left(c^{\frac{1}{n}}\right)(x-\frac{2\epsilon}{3}) - x \\ &> -\epsilon \end{aligned}$$

$\therefore\ |g_n - x| < \epsilon,$
$\therefore \lim\limits_{n\to\infty} g_n = x.$

(case 2). $x = 0,$
$\forall\ \epsilon > 0,$

the part of finding upper bound is the same.

As for lower bound, $|g_n - x| = g_n > 0 > -\epsilon$

$\therefore\ \lim\limits_{n\to\infty} g_n = x.$

(b) Is the converse true? If not, give a counter example.

Solution:

It is not true. Pick $x_{2n} = \frac{1}{2}$ and $x_{2n+1} = 2$.

x_n is divergent, while g_n converges to 1.

□

5. (a) Let $\lim_{n\to\infty} x_n = x$. Let $g_n = \frac{\sum_{i=1}^{n} x_i}{n}$, prove that $\lim_{n\to\infty} g_n = x$.

Solution:
$\because \lim_{i\to\infty} x_i = x$,

$\therefore \exists N_1$ such that $\forall i > N_1$,

$x_i = |x_i - x| < \frac{\epsilon}{2}$

take $c = \sum_{j=1}^{N_1}(x_j - x)$.

As $\lim_{k\to\infty} \frac{c}{k} = 0$,

$\therefore\ \exists N_2$, such that $\forall k > N_2$,

$\left|\frac{c}{n}\right| < \frac{\epsilon}{2}$

$\forall$ integer $n > max\{N_1, N_2\}$,

$$\begin{aligned}
|g_n - x| &= \left|\frac{\sum_{i=1}^{N_1}(x_i - x)}{n} + \frac{\sum_{i=N_1+1}^{n}(x_i - x)}{n}\right| \\
&\leq \frac{\left|\sum_{i=1}^{N_1}(x_i - x)\right|}{n} + \frac{\sum_{i=N_1+1}^{n}|x_i - x|}{n} \\
&< \frac{\epsilon}{2} + \left(\frac{n - N_1}{n}\right)\left(\frac{\epsilon}{2}\right) \\
&< \epsilon
\end{aligned}$$

$\therefore \lim_{n\to\infty} g_n = x$.

(b) Is the converse true? If not, give a counter example.
Solution:
It is not true. Pick $x_{2n} = 1$ and $x_{2n+1} = 2$.

x_n is divergent while g_n converge to $\frac{1}{2}$.

□